Free Vibrations of Plates and Beams of Pyrolytic Graphite Type Materials

CHENG-IH WU* AND JACK R. VINSON†
University of Delaware, Newark, Del.

A set of general governing equations is derived for the study of free vibrations of rectangular plates composed of a transversely isotropic material, including the effects of transverse shear deformation and rotatory inertia. Such a formulation is necessary for even geometrically thin plates, when the ratio of in-plane modulus of elasticity to transverse shear modulus is large (i.e., 20–50) which occurs in vapor deposited materials such as pyrolytic graphite, and in many fiber reinforced composite materials. The corresponding equations for beams are also derived. The case of a simply supported plate is treated in detail. Numerical results show that significant differences occur in predicting natural frequencies for various modes when the present theory is used compared to the use of classical methods for plates of these material systems. Classical methods can predict frequencies that are erroneous by a factor of nearly three.

Nomenclature

a	= plate edge length in x direction
b	= plate edge length in y direction or beam width
D	= $Eh^3/12(1-\nu^2)$, flexural rigidity of plates
E,E_c	= Young's moduli
f,k	= tracing constants
G,G_c	= shear moduli
h	= plate thickness or beam depth
L	= beam length
M_x,M_y,M_{xy}	= stress couples
Q_x,Q_y	= transverse stress resultants
r	= aspect ratio
R,R_s	= dimensionless frequencies
t	= time
T	= kinetic energy
u,v,w	= displacements in x, y, and z directions, respectively
$W(\sigma_{ij})$	= strain energy density function in terms of stresses
W_0	= amplitude of lateral displacement
α,β	= rotational displacements in x and y directions, respectively
Λ,Γ	= amplitude of in-plane displacements in x and y directions, respectively
$\epsilon_{ij},\sigma_{ij}$	= strains and stresses, respectively
λ_n,δ_m	= defined by $n\pi/a$ and $m\pi/b$, respectively
ν,ν_c	= Poisson's ratios, identity to ν_{xy} and ν_{xz} (or ν_{yz}), respectively
ρ	= mass density of materials
ψ	= Reissner's functional
ω_{mn},ω_n	= natural frequencies of plates and beams, respectively, where the motion is primarily flexural
$_s\omega_n$	= natural frequencies of beams where the motion is primarily in-plane shearing motion
$\bar{\omega}$	= frequency obtained from classical theory

Presented as Paper 69-55 at the AIAA 7th Aerospace Sciences Meeting, New York, January 20–22, 1969; submitted February 6, 1969; revision received July 10, 1969. This research was sponsored by the National Science Foundation Grant GK-684.

* Research Fellow, Department of Mechanical and Aerospace Engineering.

† Professor and Chairman, Department of Mechanical and Aerospace Engineering. Associate Fellow AIAA.

Introduction

TODAY increasing use is being made of materials systems in plate and shell construction for which conventional methods of analysis are inadequate. Among these systems are those which are transversely isotropic, and which have such a high ratio of in-plane modulus of elasticity to transverse shear modulus (E/G_c), such that even for cases in which the cross-sectional thickness h is very small compared with the smallest lateral dimension, the effects of transverse shear deformation and rotatory inertia are significant.

One such material system is pyrolytic graphite type materials. These materials are transversely isotropic, have an E/G_c ratio varying from 20 to 50, and an in-plane Poisson's ratio of -0.21.

For an isotropic beam or plate, the dynamic response according to the more accurate theory, which includes transverse shear deformation and rotatory inertia, has been studied previously.[1–4] However, none of the previous work investigated the case when transverse shear modulus is no longer equal to $E/2(1+\nu)$. In the present research the authors study the effect of unusual values of the E/G_c ratio associated with pyrolytic graphite type materials by applying Reissner's variational theorem[5] together with Hamilton's principle, as outlined by Brull and Vinson.[6]

An analytic solution is developed for the free vibration of plates made of pyrolytic graphite type materials. Hence, a set of governing equations is derived to deal with the effects of transverse shear deformation and rotatory inertia, for a plate composed of a transversely isotropic material.

Further, a simply supported plate is treated in detail for an example. A frequency equation of 6th degree polynomial is presented and a simplified equation for the primarily transverse motion is also obtained. The vibrations of beams are studied as a special case and comparisons are made between the natural frequencies determined by the methods developed herein, and those obtained by classical methods for isotropic materials for the same geometry.

Derivation of Equations

Consider the free vibration of a rectangular plate of thickness h and edge lengths a and b. The material is transversely isotropic (pyrolytic graphite for example). The plate is referred to a Cartesian coordinate system $0xyz$, the x, y plane in the middle plane of the plate and 0 at a corner of the plate. Further, a z axis is normal to the middle surface of the plate.

As in the classical plate theory, the deformation of elements in the xy plane is very small compared to the translation and the rotation, so that the displacements are of the form,

$$u(x,y,z,t) = z\alpha(x,y,t) \quad (1)$$

$$v(x,y,z,t) = z\beta(x,y,t) \quad (2)$$

$$w(x,y,z,t) = w(x,y,t) \quad (3)$$

Note that elements normal to the middle surface prior to deformation will not be assumed to remain normal to the deformed middle surface; this assumption, made in classical plate theory, is equivalent to the neglect of transverse shear deformation, and will not be made here.

Thus, for a transverse isotropic material, the relations among strains, displacements and stresses can be expressed as the following:

$$\begin{aligned} \epsilon_x &= z(\partial\alpha/\partial x) = (1/E)(\sigma_x - \nu\sigma_y) - (\nu_c/E)\sigma_z \\ \epsilon_y &= z(\partial\beta/\partial y) = (1/E)(\sigma_y - \nu\sigma_x) - (\nu_c/E)\sigma_z \\ \epsilon_{xy} &= \tfrac{1}{2}[(\partial\alpha/\partial y) + (\partial\beta/\partial x)]z = (1/2G)\sigma_{xy} \\ \epsilon_{yz} &= \tfrac{1}{2}[(\partial w/\partial y) + \beta] = (1/2G_c)\sigma_{yz} \\ \epsilon_{xz} &= \tfrac{1}{2}[(\partial w/\partial x) + \alpha] = (1/2G_c)\sigma_{xz}; \quad \epsilon_z = 0 \end{aligned} \quad (4)$$

In order to derive the equations of motion, we now apply Hamilton's principle in conjunction with Reissner's variational theorem. Thus, we may state that the motion of the plate will be such as to minimize the integral

$$\phi = \int_{t_1}^{t_2} (T - \psi)dt \quad (5)$$

where

$$\psi = \int_0^a \int_0^b \int_{-h/2}^{+h/2} [\sigma_{ij}\epsilon_{ij} - W(\sigma_{ij})]dxdydz$$

$$\sigma_{ij}\epsilon_{ij} = \sigma_x z \frac{\partial\alpha}{\partial x} + \sigma_y z \frac{\partial\beta}{\partial y} + \sigma_z \frac{\partial w}{\partial z} + \sigma_{xy}\left(\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial x}\right)z + \sigma_{yz}\left(\frac{\partial w}{\partial y} + \beta\right) + \sigma_{xz}\left(\frac{\partial w}{\partial x} + \alpha\right)$$

$$W(\sigma_{ij}) = \frac{1}{2}\left[\frac{1}{E}(\sigma_x^2 + \sigma_y^2) + \frac{1}{E_c}\sigma_z^2 - 2\left(\frac{\nu}{E}\sigma_x\sigma_y + \frac{\nu_c}{E}\sigma_y\sigma_z + \frac{\nu_c}{E}\sigma_x\sigma_z\right) + \frac{1}{G_c}(\sigma_{yz}^2 + \sigma_{xz}^2) + \frac{1}{G}\sigma_{xy}^2\right]$$

$$T = \int_0^a \int_0^b \int_{-h/2}^{+h/2} \frac{\rho}{2}\left[z^2\left(\frac{\partial\alpha}{\partial t}\right)^2 + z^2\left(\frac{\partial\beta}{\partial t}\right) + \left(\frac{\partial w}{\partial t}\right)^2\right] dxdydz$$

So far as the free vibrations are concerned, we assume the following expressions for the stresses:

$$\begin{aligned} (\sigma_x,\sigma_y,\sigma_{xy}) &= (12z/h^3)(M_x,M_y,M_{xy}) \\ (\sigma_{xz},\sigma_{yz}) &= (3/2h)\{1 - [z/(h/2)]^2\}(Q_x,Q_y) \\ \sigma_z &= 0 \end{aligned} \quad (6)$$

The substitution of Eq. (6) into (5) and the integration across the thickness give

$$\phi = \int_{t_1}^{t_2}\int_0^a\int_0^b \left\{\frac{\rho h^3}{24}\left[\left(\frac{\partial\alpha}{\partial t}\right)^2 + \left(\frac{\partial\beta}{\partial t}\right)^2\right] + \frac{\rho h}{2}\left(\frac{\partial w}{\partial t}\right)^2 - M_x\frac{\partial\alpha}{\partial x} - M_y\frac{\partial\beta}{\partial y} - M_{xy}\left(\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial x}\right) - Q_y\left(\frac{\partial w}{\partial y} + \beta\right) - Q_x\left(\frac{\partial w}{\partial x} + \alpha\right) + \frac{6M_x^2}{Eh^3} + \frac{6M_y^2}{Eh^3} + \frac{6M_{xy}^2}{Gh^3} - \frac{12\nu}{Eh^3}M_xM_y + \frac{3}{5}\frac{Q_y^3}{G_ch} + \frac{3}{5}\frac{Q_x^2}{G_ch}\right\} dtdxdy \quad (7)$$

Taking the variation of ϕ and equating it to zero, we obtain

$$\begin{aligned} \delta\phi = {} & \int_0^a\int_0^b \left[\frac{\rho h^3}{12}\left(\frac{\partial\alpha}{\partial t}\delta\alpha + \frac{\partial\beta}{\partial t}\delta\beta\right) + \rho h\frac{\partial w}{\partial t}\delta w\right]_{t_1}^{t_2} dxdy - \int_{t_1}^{t_2}\int_0^a [M_{xy}\delta\alpha + M_y\delta\beta + Q_y\delta w]_0^b\, dxdt - \int_{t_1}^{t_2}\int_0^b [M_x\delta\alpha + M_{xy}\delta\beta + Q_x\delta w]_0^a\, dtdy - {} \\ & \int_{t_1}^{t_2}\int_0^a\int_0^b \left\{\left[-\frac{\partial\alpha}{\partial x} + \frac{12}{Eh^3}M_x - \frac{12\nu}{Eh^3}M_y\right]\delta M_x + \left[-\frac{\partial\beta}{\partial y} + \frac{12}{Eh^3}M_y - \frac{12\nu}{Eh^3}M_x\right]\delta M_y + \left[-\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y} + \frac{12}{Gh^3}M_{xy}\right]\delta M_{xy} + \left[-\frac{\partial w}{\partial x} - \alpha + \frac{6}{5G_ch}Q_x\right]\delta Q_x + \left[-\frac{\partial w}{\partial y} - \beta + \frac{6}{5G_ch}Q_y\right]\delta Q_y + \right. \\ & \left[\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x - \frac{\rho h^3}{12}\frac{\partial^2\alpha}{\partial t^2}\right]\delta\alpha + \left[\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y - \frac{\rho h^3}{12}\frac{\partial^2\beta}{\partial t^2}\right]\delta\beta + \left.\left[\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \rho h\frac{\partial^2 w}{\partial t^2}\right]\delta w\right\} dtdxdy = 0 \end{aligned} \quad (8)$$

Setting the first three integrals equal to zero, the natural boundary conditions are found and the initial conditions which must be specified. Since the variations δM_x, δM_y, δM_{xy}, δQ_x, δQ_y, $\delta\alpha$, $\delta\beta$, and δw are all independent arbitrary functions of x and y, the only way in which the definite integral of Eq. (8) can be made to vanish is by requiring the unknowns M_x, M_y, M_{xy}, Q_x, Q_y, α, β, and w to satisfy the following equations:

$$(\partial M_x/\partial x) + (\partial M_{xy}/\partial y) - Q_x - (\rho h^3 f/12)\partial^2\alpha/\partial t^2 = 0 \quad (9)$$

$$(\partial M_{xy}/\partial x) + (\partial M_y/\partial y) - Q_y - (\rho h^3 f/12)\partial^2\beta/\partial t^2 = 0 \quad (10)$$

$$(\partial Q_x/\partial x) + (\partial Q_y/\partial y) - \rho h(\partial^2 w/\partial t^2) = 0 \quad (11)$$

$$(\partial\alpha/\partial x) - (12/Eh^3)M_x + (12\nu/Eh^3)M_y = 0 \quad (12)$$

$$(\partial\beta/\partial y) - (12/Eh^3)M_y + (12\nu/Eh^3)M_x = 0 \quad (13)$$

$$(\partial\beta/\partial x) + (\partial\alpha/\partial y) - (12/Gh^3)M_{xy} = 0 \quad (14)$$

$$(\partial w/\partial x) + \alpha - (6k/5G_ch)Q_x = 0 \quad (15)$$

$$(\partial w/\partial y) + \beta - (6k/5G_ch)Q_y = 0 \quad (16)$$

In the previous equations, we have introduced two tracing constants f and k for the purpose of identifying terms. We note that Eqs. (9) and (10) are identical to the corresponding moment equilibrium condition of classical plate theory, except for the terms $(\rho h^3 f/12)\partial^2\alpha/\partial t^2$ and $(\rho h^3 f/12)\ \partial^2\beta/\partial t^2$ which represent the contribution of rotatory inertia. Thus, when we set $f = 1$ in the resulting solutions, we include the effects of rotatory inertia, and when we set $f = 0$, we obtain a theory which neglects these effects. Equation (11) is identical to the classical plate theory equation for transverse force equilibrium. Equations (12–14) are identical to classical theory, moment-curvature relations. In Eqs. (15) and (16), the terms $6Q_xk/5G_ch$ and $6Q_yk/5G_ch$ represent the effects of transverse shear deformation which are introduced when $k = 1$ and neglected when $k = 0$.

It is convenient to reduce the Euler-Lagrange equations (9–16) to a system of three equations in the unknown dis-

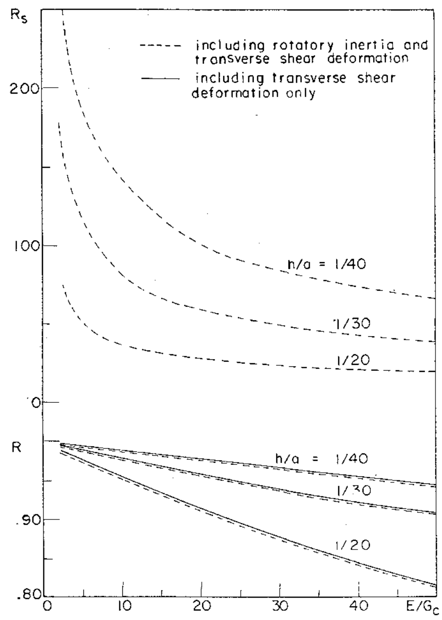

Fig. 1 Dimensionless frequency ($n = 1$) vs E/G_c for simply supported beams.

placements w, α, and β. The results are

$$\alpha + \frac{\partial w}{\partial x} - \frac{6kD}{5G_c h}\left(\frac{\partial^2\alpha}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2\alpha}{\partial y^2} - \frac{\rho f(1-\nu^2)}{E}\frac{\partial^2\alpha}{\partial t^2} + \frac{1+\nu}{2}\frac{\partial^2\beta}{\partial x \partial y}\right) = 0$$

$$\beta + \frac{\partial w}{\partial y} - \frac{6kD}{5G_c h}\left(\frac{\partial^2\beta}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2\beta}{\partial x^2} - \frac{\rho f(1-\nu^2)}{E}\frac{\partial^2\beta}{\partial t^2} + \frac{1+\nu}{2}\frac{\partial^2\alpha}{\partial x \partial y}\right) = 0 \qquad (17)$$

$$\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial y} + \nabla^2 w - \frac{6k\rho}{5G_c}\frac{\partial^2 w}{\partial t^2} = 0$$

We now consider a simply supported plate. The boundary conditions are

$$w(0,y) = w(a,y) = 0,\ w(x,0) = w(x,b) = 0$$
$$M_x(0,y) = M_x(a,y) = 0,\ M_y(x,0) = M_y(x,b) = 0 \qquad (18)$$
$$\beta(0,y) = \beta(a,y) = 0,\ \alpha(x,0) = \alpha(x,b) = 0$$

When the plate is oscillating in a normal mode, the motion is harmonic such that the solutions for α, β, and w may be taken in the form

$$w = W_0 \sin(n\pi x/a)\sin(m\pi y/b)\cos\omega_{mn}t$$
$$\alpha = \Gamma\cos(n\pi x/a)\sin(m\pi y/b)\cos\omega_{mn}t \qquad (19)$$
$$\beta = \Lambda\sin(n\pi x/z)\cos(m\pi y/b)\cos\omega_{mn}t$$

It should be noted that the expressions of (19) satisfy the boundary conditions and specify the initial conditions. The substitution of Eqs. (19) in Eqs. (17) yields three simultaneous homogeneous algebraic equations for the amplitudes W_0, Γ, and Λ; these are

$$\lambda_n W_0 + \left\{1 + \frac{6Dk}{5G_c h}\left[\lambda_n^2 + \frac{1-\nu}{2}\delta_m^2 - \frac{\rho f(1-\nu^2)}{E}\omega_{mn}^2\right]\right\}\Gamma + \frac{3D(1+\nu)k}{5G_c h}\lambda_n\delta_m\Lambda = 0$$

$$\delta_m W_0 + \frac{3D(1+\nu)k}{5G_c h}\lambda_n\delta_m\Gamma + \left\{1 + \frac{6Dk}{5G_c h}\times\left[\delta_m^2 + \frac{1-\nu}{2}\lambda_n^2 - \frac{\rho f(1-\nu^2)}{E}\omega_{mn}^2\right]\right\}\Lambda = 0 \qquad (20)$$

$$\left[\lambda_n^2 + \delta_m^2 - \frac{6\rho k}{5G_c}\omega_{mn}^2\right]W_0 + \lambda_n\Gamma + \delta_m\Lambda = 0$$

where $\lambda_n = n\pi/a$ and $\delta_m = m\pi/b$.

Since Eqs. (20) form a homogeneous system, the condition for a nontrivial solution is that the determinant of the coefficients of W_0, Γ, and Λ vanishes; this yields the frequency equation as the following:

$$\omega_{mn}^6 - \left[\frac{E(3-\nu)}{2f(1-\nu^2)}(\lambda_n^2+\delta_m^2) + \left(\frac{5G_c}{6k\rho}\right)(\lambda_n^2+\delta_m^2) + \left(\frac{5G_c h}{6kD}\right)\frac{2E}{\rho f(1-\nu^2)}\right]\omega_{mn}^4 + \left[\frac{1-\nu}{2}\frac{E^2}{\rho^2 f^2(1-\nu^2)^2}\times(\lambda_n^2+\delta_m^2)^2 + \left(\frac{5G_c}{6k\rho}\right)\frac{E(3-\nu)}{2\rho f(1-\nu^2)}(\lambda_n^2+\delta_m^2)^2 + \left(\frac{5G_c h}{6Dk}\right)\left(\frac{3-\nu}{2}\right)\frac{E^2}{\rho^2 f^2(1-\nu^2)^2}(\lambda_n^2+\delta_m^2) + \left(\frac{5G_c}{6\rho k}\right)\left(\frac{5G_c h}{6Dk}\right)\frac{E}{\rho f(1-\nu^2)}(\lambda_n^2+\delta_m^2) + \left(\frac{5G_c h}{6Dk}\right)^2\frac{E^2}{\rho^2 f^2(1-\nu^2)^2}\right]\omega_{mn}^2 - \left[\left(\frac{5G_c}{6\rho k}\right)\left(\frac{1-\nu}{2}\right)\times\frac{E^2}{\rho^2 f^2(1-\nu^2)^2}(\lambda_n^2+\delta_m^2)^3 + \left(\frac{5G_c}{6Dk}\right)\left(\frac{5G_c}{6\rho k}\right)\times\frac{E^2}{\rho^2 f^2(1-\nu^2)^2}(\lambda_n^2+\delta_m^2)^2\right] = 0 \qquad (21)$$

The natural frequencies of the case where both transverse shear deformation and rotatory inertia are included may be obtained by solving Eq. (21) with $k = f = 1$. The frequency equation is then of the form

$$\omega_{mn}^6 - \left(\frac{E}{\rho h^2}\right)\left[\frac{(3-\nu)}{2(1-\nu^2)}(\lambda_n^2+\delta_m^2)h^2 + \frac{5}{6}\left(\frac{G_c}{E}\right)\times(\lambda_n^2+\delta_m^2)h^2 + 20\left(\frac{G_c}{E}\right)\right]\omega_{mn}^4 + \left(\frac{E}{\rho h^2}\right)^2\times\left[\frac{(1-\nu)}{2(1-\nu^2)^2}(\lambda_n^2+\delta_m^2)^2h^4 + \frac{5(3-\nu)}{12(1-\nu^2)}\left(\frac{G_c}{E}\right)\times(\lambda_n^2+\delta_m^2)^2h^4 + \frac{5(3-\nu)}{1-\nu^2}\left(\frac{G_c}{E}\right)(\lambda_n^2+\delta_m^2)h^2 + \frac{25}{3}\left(\frac{G_c}{E}\right)^2(\lambda_n^2+\delta_m^2)h^2 + 100\left(\frac{G_c}{E}\right)^2\right]\omega_{mn}^2 - \left(\frac{E}{\rho h^2}\right)^3\left[\frac{5(1-\nu)}{12(1-\nu^2)^2}\left(\frac{G_c}{E}\right)(\lambda_n^2+\delta_m^2)^3h^6 + \frac{25}{3(1-\nu^2)}\left(\frac{G_c}{E}\right)^2(\lambda_n^2+\delta_m^2)^2h^4\right] = 0 \qquad (22)$$

To obtain a simplified theory neglecting the effects of rotatory inertia, but retaining transverse shear deformation, we set $k = 1$ and $f = 0$ after multiplying the frequency equation (21) by f^2, the resulting simplified frequency equation may be rewritten as

$$\omega_{mn}^2 = \frac{(D/\rho h)(\lambda_n^2+\delta_m^2)^2}{1 + [h^2/10(1-\nu^2)](E/G_c)(\lambda_n^2+\delta_m^2)} \qquad (23)$$

Finally, to obtain a frequency equation in which both transverse shear deformation and rotatory inertia are neglected, we multiply Eq. (21) by $k^2 f^2$ and set $k = f = 0$; the

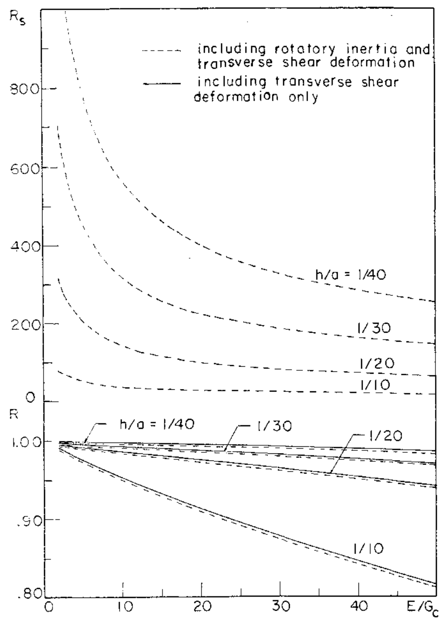

Fig. 2 Dimensionless freqency (n = 2) vs E/G_c for simply supported beams.

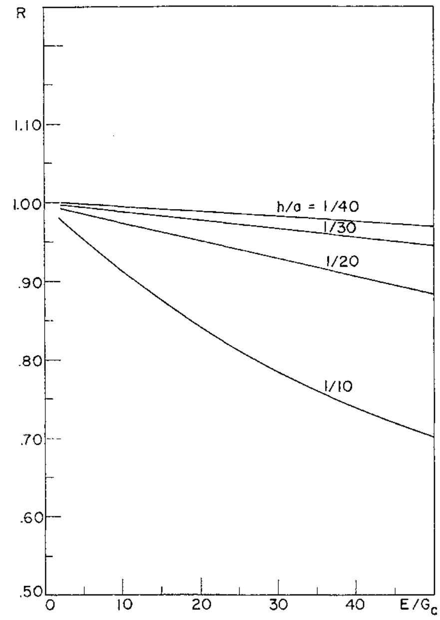

Fig. 4 Dimensionless frequency ($n = m = 1$) vs E/G_c for simply supported plates, $r = 1$.

frequency is then given by

$$\bar{\omega}_{mn}^2 = (D/\rho h)(\lambda_n^2 + \delta_m^2)^2 \tag{24}$$

the well-known solution of classical plate theory for a simply supported plate.

The free vibrations of a beam are studied as a special case. The general frequency equation of a simply supported beam is obtained from the corresponding plate equations (20) by letting $\nu = \Lambda = \delta_m = 0$. Setting $k = f = 1$ both transverse shear deformation and rotatory inertia are included in the following equation:

$$\omega_n^4 - (E/\rho h^2)[10(G_c/E) + n^2\pi^2(h/L)^2 \times \{1 + (\tfrac{5}{6}\cdot G_c/E)\}]\omega_n^2 + \tfrac{5}{6}(E/\rho h^2)^2(G_c/E)n^4\pi^4(h/L)^4 = 0 \tag{25}$$

If we set $k = 1$ and $f = 0$, accordingly the resulting frequency equation, neglecting rotatory inertia but retaining transverse shear deformation is written as

$$\omega_n^2 = \frac{(EI/\rho A)(n\pi/L)^4}{1 + (n^2\pi^2/10)(E/G_c)(h/L)^2} \tag{26}$$

where $I = bh^3/12$ and $A = bh$.

Similarly, the frequency equation of classical beam theory, neglecting the effects listed above ($K = f = 0$), for a simply supported beam is obtained;

$$\bar{\omega}_n^2 = (EI/\rho A)(n\pi/L)^4 \tag{27}$$

Numerical Results

In order to show the errors that result in the use of classical isotropic beam and plate theory to describe the vibrational behavior of beams and plates of pyrolytic graphite type materials, curves have been plotted using the ratio of the present plate or beam frequency equations to the correspond-

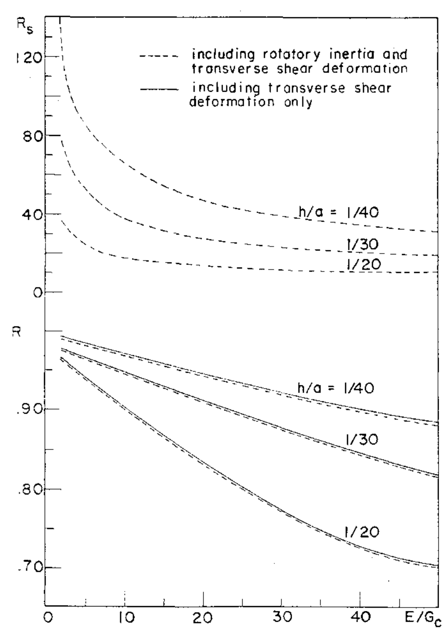

Fig. 3 Dimensionless frequency (n = 3) vs E/G_c for simply supported beams.

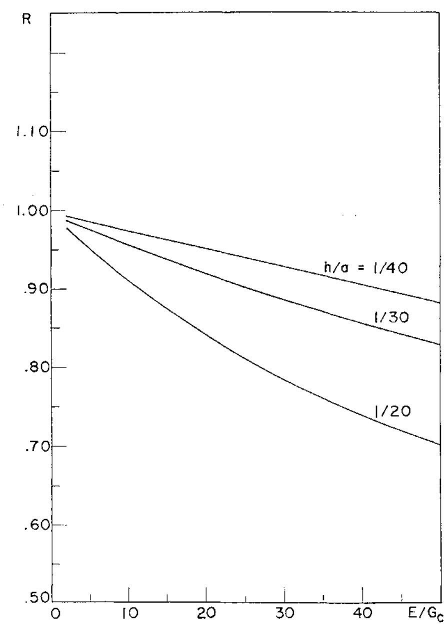

Fig. 5 Dimensionless frequency ($n = m = 2$) vs E/G_c for simply supported plates, $r = 1$.

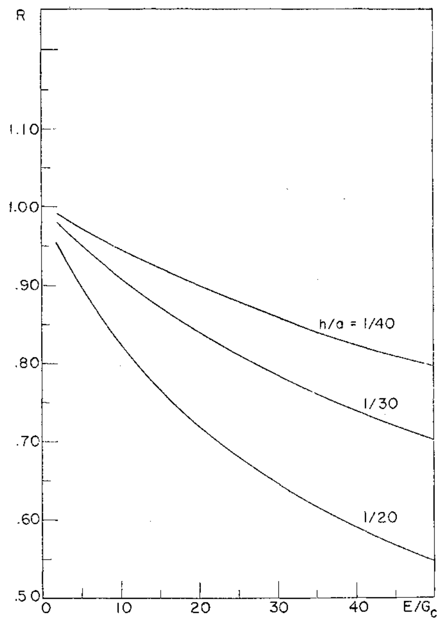

Fig. 6 Dimensionless frequency ($n = m = 3$) vs E/G_c for simply supported plates, $r = 1$.

ing classical theory equations as a function of the value of E/G_c over the range of interest. In the case of plate calculations $\nu = -0.21$ is used, since this is the value for pyrolytic graphite. Geometrically, for both beams and plates dimensional ratios have been taken which are considered as thin for plates or beams of isotropic materials. The curves of Figs. 1–9 are given for the fundamental, the second and the third mode free vibration of small amplitude of simply supported plates as well as beams. These curves are specified in three classes of frequency ratios as the following: 1) Including the effects of both transverse shear deformation and rotatory inertia in beams as shown in Figs. 1–3. Here, the dimensionless frequency ratios are given by

$$R_s^2 = s\omega_n^2/\bar{\omega}_n^2, \quad R^2 = \omega_n^2/\bar{\omega}_n^2 \tag{28}$$

where

$$s\omega_n^2 = \tfrac{1}{2}[k_1 + (k_1^2 - 4k_2)^{1/2}] = \text{frequency of primarily shear vibrations} \tag{29}$$

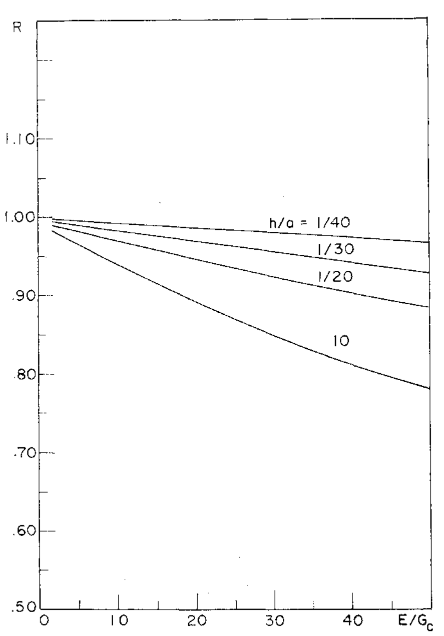

Fig. 7 Dimensionless frequency ($n = m = 1$) vs E/G_c for simply supported plates, $r = 2.0$.

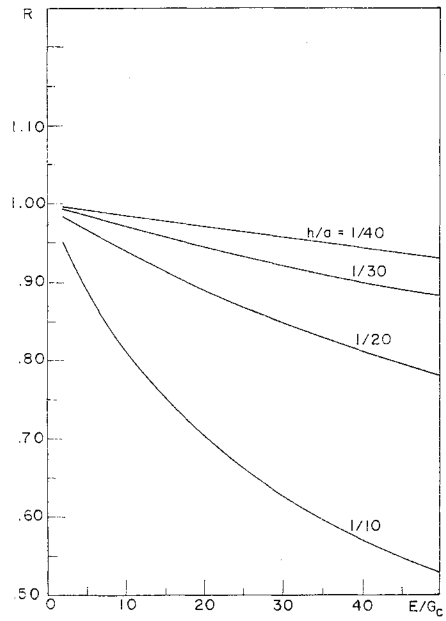

Fig. 8 Dimensionless frequency ($n = m = 2$) vs E/G_c for simply supported plates, $r = 2.0$.

$$\omega_n^2 = \tfrac{1}{2}[k_1 - (k_1^2 - 4k_2)^{1/2}] = \text{frequency of primarily flexural vibrations} \tag{30}$$

$$k_1 = (E/\rho h^2)[10 + \tfrac{5}{6}n^2\pi^2(h/L)^2](G_c/E) + n^2\pi^2(h/L)^2$$

$$k_2 = (E/\rho h^2)^2 \cdot \tfrac{5}{6}n^4\pi^4(h/L)^4(G_c/E)$$

2) The frequency, neglecting rotatory inertia but involving transverse shear deformation in the beam as shown in Figs. 1–3, where

$$R^2 = \omega_n^2/\bar{\omega}_n^2 = 1/1 + (n^2\pi^2/10)(E/G_c)(h/L)^2 \tag{31}$$

3) The frequency including transverse shear deformation only for a simply supported plate and assuming the number of vibration modes are the same in x direction as well as in y direction namely $m = n$ as shown in Figs. 4–9.

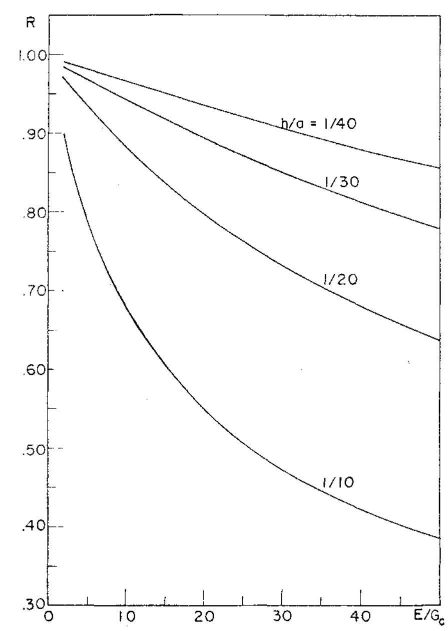

Fig. 9 Dimensionless frequency ($n = m = 3$) vs E/G_c for simply supported plates, $r = 2.0$.

The dimensionless frequency ratio is given by

$$R^2 = \frac{\omega_{nn}^2}{\bar{\omega}_{nn}^2} = \frac{1}{1 + [n^2\pi^2/10(1 - \nu^2)](E/G_c)(h/a)^2(1 + 1/r^2)} \quad (32)$$

It should be noted that the dimensionless frequencies are functions of n, h/a, r, and E/G_c but independent of density of the material. The factor $D/\rho h$ if Eq. (23), $E/\rho h^2$ in Eq. (25), and $EI/\rho A$ in Eq. (26) will automatically be eliminated when taking the ratio to the classical Eqs. (24) and (27).

Discussion and Conclusion

The frequency curves for plates show that the effect of transverse shear deformation is very important for thin plates of materials having a large E/G_c ratio. For example, in the mode $m = n = 3$, with thickness to length ratio of $\frac{1}{10}$, aspect ratio 2, and $E/G_c = 50$, the actual natural frequency is 38% of that predicted by classical plate theory.

For beams, the effect of rotatory inertia is so small (<1%) in the primarily flexural vibration, that for the mode which is composed primarily of transverse motion, the frequency equations which neglect rotatory inertia and retain transverse shear deformation can be used for the advantage of simplicity.

However, the rotatory inertia term identifies natural frequencies associated with modes primarily composed of in-plane shear deformations. These are associated with significantly higher frequencies for a given wave number. These modes and frequencies cannot be obtained from the simplified expressions discussed previously.

When calculating the natural frequencies of vibrations or investigating the forced vibration of thin plates or beams, one cannot use conventional methods of analysis based upon isotropic elastic theory if the plate or beam is composed of a pyrolytic graphite type material in which $20 \leq E/G_c \leq 50$.

By simply making the material properties isotropic, the methods developed herein can be employed for the study of the vibrations of isotropic plates and beams, including effects of transverse shear deformation and rotatory inertia.

The present methods apply also to the dynamic response of plates and beams of the class of *composite materials* which have $E_x = E_y$ and $\nu_{xy} = \nu_{yx}$.

References

[1] Mindlin, R. D., "Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic Elastic Plates," *Journal of Applied Mechanics*, Vol. 18, 1951, pp. 31–38.

[2] Mindlin, R. D., "Flexural Vibrations of Rectangular Plates," *Journal of Applied Mechanics*, Vol. 23, 1956, pp. 430–436.

[3] Miklowitz, J., "Flexural Stress Waves in an Infinite Elastic Plate Due to Suddenly Applied Concentrated Transverse Load," *Journal of Applied Mechanics*, Vol. 27, 1960, pp. 681–689.

[4] Miklowitz, J., "Wave Propagation in an Elastic Beam or Plate on an Elastic Foundation," *Journal of Applied Mechanics*, Vol. 30, 1962, pp. 459–464.

[5] Reissner, E., "On a Variational Theorem in Elasticity," *Journal of Mathematics and Physics*, Vol. 29, 1950, pp. 90–95.

[6] Brull, M. A. and Vinson, J. R., "On Reissner's Variational Principle and Its Application to the Theory of Moderately Thick Beams," 1962, Dyna/Structures Inc.

[7] Wu, C. I., "On the Lateral Vibration of Rectangular Plates of Pyrolytic Graphite Type Materials," MS thesis, 1968, Dept. of Mechanical and Aerospace Engineering, Univ. of Delaware, Newark, Del.